

Parasupersymmetric Quantum Mechanics of Order 3 and a Generalized Witten Index

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Abstract

In this paper we generalize the Rubakov-Spiridonov parasupersymmetry algebra to the order 3 case. We also generalize the notion of the Witten index, and we provide a class of models satisfying our parasupersymmetry algebra. Finally, we show that there is a correspondence between the Hamiltonian and the index in our class of models.

1 Introduction

The notion of supersymmetry - symmetry between bosonic and fermionic degrees of freedom - is an idea that is believed to play an important role in physics. For an excellent review, see [3]. However, it is not clear that nature is supersymmetric at the level of elementary particles. Besides the ordinary Bose-Einstein and Fermi-Dirac statistics there exists an intermediate set of para-Fermi generalizations [5], characterized by an integer p , indicating the maximum number of particles which may occupy the same state. Ordinary fermions correspond to $p = 1$ and bosons to $p = \infty$. The particular case $p = 3$ is especially interesting since it is an alternative way of describing quarks (without using colours) - see [4]. The natural generalization of supersymmetry - so-called parasupersymmetry (abbreviated to parasusy) - is a symmetry between bosonic and parafermionic degrees of freedom (more precisely, between “neighbouring” parafermion levels, taking the bosonic level to be parafermion level 0), and one might hope that it plays a similarly important role to supersymmetry in the description of nature. The parasupersymmetric structure will of course depend on the number, p , of parafermions that can occupy the same state.

Supersymmetric quantum mechanics, the $p = 1$ case, was studied by Witten in [8]. The generalization of this to the $p = 2$ case was done by Rubakov and Spiridonov in [7], and also in a slightly different way by Beckers and Debergh in [1]. In this paper we present the generalization of these results to the $p = 3$ case. We give a natural form of the parasupersymmetry algebra and present a class of models satisfying it. Also, we will discuss the generalization proposed in [6] of the Witten index [8] in this case, and relate the index to the spectra of the Hamiltonian in our $p = 3$ model, as well as describing the same correspondence in the cases $p = 1$ and $p = 2$.

The paper is organized as follows: in section 2 we recall some basic features of the $p = 1$ and $p = 2$ cases. In section 3 we give the definition of the generalized $p = 3$ parasupersymmetry algebra, together with the definition of generic families and the stratification of the Hilbert space into a direct sum of certain low-dimensional invariant subspaces. In section 4 we recall the definition [6] of the parasupersymmetry index for $p = 3$, generalizing the Witten index, and prove a formula for its calculation. In section 5, we generalize the class of models of section 2 to the $p = 3$ case, and prove that it satisfies the relations of our parasusy algebra from section 3. Finally, in section 6, we prove that the spectrum of the Hamiltonian of our model is generically 4-fold degenerate and positive, except possibly at the lowest (vacuum) levels, and furthermore, that there is a correspondence between the index of the model and the possible forms of the spectrum. We also present the same results (correspondence between the index and the spectrum) for the models in the cases $p = 1$ and $p = 2$.

2 The Cases $p = 1$ and $p = 2$

Let us review briefly the supersymmetric case $p = 1$. We will denote the bosonic coordinate by x , and the operator of taking the x -derivative by d_x (we avoid using the momentum operator explicitly, so as not to have to introduce the imaginary unit i into the expressions).

One can introduce supersymmetry operators, also known as supercharges, by:

$$Q = \begin{pmatrix} 0 & 0 \\ -d_x + W(x) & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & d_x + W(x) \\ 0 & 0 \end{pmatrix}, \quad (1)$$

where $W(x)$ is an arbitrary function (of class \mathcal{C}^2), known as the superpotential. The supercharges obey the following algebra:

$$\begin{aligned} Q^2 = Q^{\dagger 2} = [H, Q] = [H, Q^\dagger] &= 0, \\ QQ^\dagger + Q^\dagger Q &= 2H, \end{aligned} \quad (2)$$

where the Hamiltonian is given by:

$$H = \frac{1}{2}(-d_x^2 + W^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} W' & 0 \\ 0 & -W' \end{pmatrix}.$$

The Hamiltonian and (conserved) supercharges act on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^2) = \left\{ \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} \mid f_i \in L^2(\mathbb{R}, \mathbb{R}) \right\}$$

with the natural inner product coming from the L^2 inner product and the standard inner product on \mathbb{R}^2 . If we define \mathcal{H}_1 to be the subspace with $f_0 = 0$ and \mathcal{H}_0 to be the subspace with $f_1 = 0$, then we have the following stratification:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 \oplus \mathcal{H}_1, \quad H(\mathcal{H}_i) \subset \mathcal{H}_i, \\ Q(\mathcal{H}_1) &\subset \mathcal{H}_0, \quad Q(\mathcal{H}_0) = \{0\}, \\ Q^\dagger(\mathcal{H}_0) &\subset \mathcal{H}_1, \quad Q^\dagger(\mathcal{H}_1) = \{0\}. \end{aligned}$$

Following [7] the simplest generalization of (1) to the $p = 2$ case is given by the following parasusy operators:

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ -d_x + W_1 & 0 & 0 \\ 0 & -d_x + W_2 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & d_x + W_1 & 0 \\ 0 & 0 & d_x + W_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3)$$

where $W_1(x)$ and $W_2(x)$ are functions (of class \mathcal{C}^2) whose properties are still to be specified. We only have to find the Hamiltonian H in order to obtain a suitable algebra generated by H , Q and Q^\dagger . This time, bilinear combinations of Q and Q^\dagger cannot have the form $-d_x^2 \mathbb{I}_3 + U(x)$ with $U(x)$ a 3×3 matrix depending only on the coordinate x . However, the trilinear combinations satisfy:

$$\begin{aligned} Q^3 = Q^{\dagger 3} = [H, Q] = [H, Q^\dagger] &= 0, \\ Q^2 Q^\dagger + Q Q^\dagger Q + Q^\dagger Q^2 &= 4QH, \\ Q^{\dagger 2} Q + Q^\dagger Q Q^\dagger + Q Q^{\dagger 2} &= 4Q^\dagger H, \end{aligned} \quad (4)$$

where the Hamiltonian is given by:

$$4H = (-2d_x^2 + W_1^2 + W_2^2)\mathbb{I}_3 + \text{diag}(3W_1' + W_2', -W_1' + W_2', -W_1' - 3W_2'),$$

provided that the superpotentials satisfy:

$$(W_2^2 - W_1^2)' + W_2'' + W_1'' = 0. \quad (5)$$

Here all the operators are acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^3)$ with its canonical inner product, and we have a stratification:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2, & H(\mathcal{H}_j) &\subset \mathcal{H}_j, \\ Q(\mathcal{H}_2) &\subset \mathcal{H}_1, & Q(\mathcal{H}_1) &\subset \mathcal{H}_0, & Q(\mathcal{H}_0) &= \{0\}, \\ Q^\dagger(\mathcal{H}_0) &\subset \mathcal{H}_1, & Q^\dagger(\mathcal{H}_1) &\subset \mathcal{H}_2, & Q^\dagger(\mathcal{H}_2) &= \{0\}. \end{aligned}$$

In both cases $p = 1$ and $p = 2$, we can introduce the (para)fermion number operator N , which on each \mathcal{H}_j is given by multiplication by j and on the rest of \mathcal{H} by linear extension. In the supersymmetric case it coincides with the fermion number operator F used in [8].

Now, even if we forget about the specific forms (1) and (3) and work just with the respective parasupersymmetry algebras (2) and (4), we can find further structures and a finer stratification of the Hilbert space. For the case $p = 1$ this was analyzed in [8] and for the case $p = 2$ in [6]. In both cases it can be shown that the operators $H, QQ^\dagger, Q^\dagger Q$ and N commute, and that, if Ψ is an eigenstate of all four operators then so are $Q\Psi$ and $Q^\dagger\Psi$. Thus we can define the family $\mathcal{F}(\Psi)$ corresponding to such a state, as being the subspace spanned by Ψ and all the states that can be obtained from Ψ by acting on it with Q 's and Q^\dagger 's. Such a family will be an invariant subspace for each of the operators H, Q, Q^\dagger and N . Also, the dimension of each such family is less than or equal to $p + 1$, and if it is equal to $p + 1$ we will call such a family generic. Finally, with suitable restrictions on the Hamiltonian (having a discrete spectrum, being bounded below and with a finite multiplicity for each eigenvalue) it can be proved that there exists a sequence $\Psi_n \in \mathcal{H}$ such that:

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{F}(\Psi_n).$$

Further details for the cases $p = 1$ and $p = 2$ can be found in [8] and [6].

In the next section we will introduce a natural generalization of the algebras (2) and (4) to the $p = 3$ case and prove the analogues of all the above statements.

3 The $p = 3$ Parasupersymmetry Algebra

As indicated in the previous section, the algebraic relations (2) and (4) satisfied by the operators give rise to a decomposition of the Hilbert space into natural finite-dimensional subspaces. This statement holds irrespective of any particular model obeying the algebra. We will now develop the same algebraic perspective for the case $p = 3$, starting with the following definition.

Definition 1 Let \mathcal{H} be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and H a self-adjoint Hamiltonian operator on \mathcal{H} with purely discrete spectrum. (\mathcal{H}, H) will be said to be parasymmetric of order $p = 3$ iff the following three conditions hold:

1. $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, $H(\mathcal{H}_j) \subset \mathcal{H}_j$
2. There exists a densely defined operator Q on \mathcal{H} , with adjoint Q^\dagger such that:

$$Q(\mathcal{H}_0) = 0, \quad Q(\mathcal{H}_j) \subset \mathcal{H}_{j-1}, \quad j = 1, 2, 3$$

$$Q^\dagger(\mathcal{H}_3) = 0, \quad Q^\dagger(\mathcal{H}_{j-1}) \subset \mathcal{H}_j, \quad j = 1, 2, 3$$

3. The operators H, Q, Q^\dagger satisfy the following parasusy algebra:

$$\begin{aligned} (i) \quad & [Q, H] = [Q^\dagger, H] = 0 \\ (ii) \quad & Q^3 Q^\dagger + Q^2 Q^\dagger Q + Q Q^\dagger Q^2 + Q^\dagger Q^3 = 6Q^2 H \\ (iii) \quad & Q Q^\dagger{}^2 Q = Q^\dagger Q^2 Q^\dagger \\ (iv) \quad & Q^\dagger{}^3 Q + Q^\dagger{}^2 Q Q^\dagger + Q^\dagger Q Q^\dagger{}^2 + Q Q^\dagger{}^3 = 6Q^\dagger{}^2 H \end{aligned}$$

Remark 1 This algebra generalizes the Rubakov and Spiridonov [7] algebra of equation (4), and it is partly of the form suggested in [6]. Relation (iv) is included just for the sake of symmetry since it is the adjoint of (ii). Also, relation (ii) is symmetric in the following sense: we have all four quadrilinear combinations with one Q and three Q^\dagger 's with the same coefficient (equal to 1) on the left hand side, and the right hand side has the same homogeneity as the left hand side if H is replaced by $Q Q^\dagger$ or $Q^\dagger Q$, analogously to the $p = 1$ and $p = 2$ algebras (2) and (4). The constant on the right hand side of relations (ii) and (iv) is arbitrary, but we choose it to be 6, since for the model in section 5 this choice leads to a Hamiltonian with the standard factor $-1/2$ multiplying d_x^2 in the kinetic term. Also, for the decomposition into families that we prove below, we need the operators $Q Q^\dagger$, $Q^\dagger Q$ to commute, i.e. relation (iii). Note, that this is an expression involving two quadrilinear combinations with two Q 's and two Q^\dagger 's, but is much simpler than the form suggested in [6]. Thus the relations (i) – (iv) are, in fact, the “minimal” set of conditions required in order to have the fine stratification properties that we are looking for. Finally, the algebra is satisfied by the natural $p = 3$ generalization of the class of models from section 2, as will be shown in section 5.

The parafermion number operator N is given on the \mathcal{H}_j 's by:

$$N(f) = jf, \quad f \in \mathcal{H}_j,$$

and on the whole of \mathcal{H} by linear extension.

Proposition 1 The operators $H, Q Q^\dagger, Q^\dagger Q, N, Q^2 Q^\dagger{}^2, Q^\dagger{}^2 Q^2, Q Q^\dagger{}^2 Q, Q^\dagger Q^2 Q^\dagger, Q Q^\dagger Q Q^\dagger$ and $Q^\dagger Q Q^\dagger Q$ commute.

Proof: Since the last four operators are products of QQ^\dagger and $Q^\dagger Q$, the only nontrivial statements are the commutativity of QQ^\dagger , $Q^\dagger Q$, $Q^2Q^{\dagger 2}$ and $Q^{\dagger 2}Q^2$. We have:

$$\begin{aligned}
[Q^2Q^{\dagger 2}, QQ^\dagger] &= Q^2Q^{\dagger 2}QQ^\dagger - QQ^\dagger Q^2Q^{\dagger 2} = Q(QQ^{\dagger 2}Q - Q^\dagger Q^2Q^\dagger)Q^\dagger \\
&= Q[QQ^\dagger, Q^\dagger Q]Q^\dagger = 0 \\
[Q^2Q^{\dagger 2}, Q^\dagger Q] &= Q^2Q^{\dagger 3}Q - Q^\dagger Q^3Q^{\dagger 2} \\
&= Q^2(6Q^{\dagger 2}H - Q^{\dagger 2}QQ^\dagger - Q^\dagger QQ^{\dagger 2} - QQ^{\dagger 3}) - \\
&\quad -(6Q^2H - Q^3Q^\dagger - Q^2Q^\dagger Q - QQ^\dagger Q^2)Q^{\dagger 2} \\
&= 6Q^2Q^{\dagger 2}H - Q^2Q^{\dagger 2}QQ^\dagger - Q^2Q^\dagger QQ^{\dagger 2} - Q^3Q^{\dagger 3} - \\
&\quad -6Q^2Q^{\dagger 2}H + Q^3Q^{\dagger 3} + Q^2Q^\dagger QQ^{\dagger 2} + QQ^\dagger Q^2Q^{\dagger 2} \\
&= [QQ^\dagger, Q^2Q^{\dagger 2}] = 0
\end{aligned}$$

In proving the second commutation relation, in the second equality we used relations (ii) and (iv) from the parasusy algebra. In an analogous manner we can obtain the commutativity of $Q^{\dagger 2}Q^2$ and QQ^\dagger or $Q^\dagger Q$. The commutativity of $Q^2Q^{\dagger 2}$ and $Q^{\dagger 2}Q^2$ is trivial since we have $Q^4 = Q^{\dagger 4} = 0$. ■

From now on, we will denote the commuting set of operators in Proposition 1 by S . Thus, by restricting to any finite dimensional subspace of \mathcal{H} , the operators in S can be simultaneously diagonalized.

Proposition 2 *If Ψ is an eigenstate of all the operators belonging to S , then so are $Q\Psi$ and $Q^\dagger\Psi$.*

Proof: The statement is obvious for H and N . For QQ^\dagger , the nontrivial part is for $Q^\dagger\Psi$. Since Ψ is an eigenstate of QQ^\dagger , we have $QQ^\dagger\Psi = c\Psi$, for some scalar c . If $c = 0$, then $\|Q^\dagger\Psi\|^2 = \langle Q^\dagger\Psi, Q^\dagger\Psi \rangle = \langle \Psi, QQ^\dagger\Psi \rangle = 0$, i.e. $Q^\dagger\Psi = 0$ (and so $QQ^\dagger Q^\dagger\Psi = 0 = Q^\dagger\Psi$). If $c \neq 0$ then we have:

$$QQ^{\dagger 2}\Psi = (1/c)QQ^{\dagger 2}QQ^\dagger\Psi \stackrel{(iii)}{=} (1/c)Q^\dagger Q^2Q^{\dagger 2}\Psi = (1/c)dQ^\dagger\Psi,$$

where d is the eigenvalue of Ψ with respect to $Q^2Q^{\dagger 2}$. The reasoning is analogous for $Q^\dagger Q(Q\Psi)$. For “higher” cases, the nontrivial parts are proving that $Q^2Q^{\dagger 2}Q^\dagger\Psi$ and $Q^{\dagger 2}Q^2Q\Psi$ are proportional to $Q^\dagger\Psi$ and $Q\Psi$, respectively. In the first case, we have:

$$\begin{aligned}
Q^2Q^{\dagger 2}Q^\dagger\Psi &\stackrel{(iv)}{=} 6QQ^{\dagger 2}H\Psi - QQ^{\dagger 2}Q^\dagger Q\Psi - QQ^{\dagger 2}QQ^\dagger\Psi - QQ^\dagger QQ^{\dagger 2}\Psi \\
&= 6(QQ^\dagger)H(Q^\dagger\Psi) - (QQ^\dagger)Q^\dagger(Q^\dagger Q)\Psi - (QQ^\dagger)(Q^\dagger Q)Q^\dagger\Psi - (QQ^\dagger)^2Q^\dagger\Psi.
\end{aligned}$$

Now, all terms on the RHS are proportional to $Q^\dagger\Psi$ because of the first part of the proof. Again the reasoning is analogous for $Q^{\dagger 2}Q^3\Psi$. ■

Definition 2 *Given an eigenstate Ψ of S , the family $\mathcal{F}(\Psi)$ is the subspace of \mathcal{H} spanned by Ψ and all the states obtained by operating on Ψ with Q ’s and Q^\dagger ’s.*

Proposition 3 For all $\Psi \in \mathcal{H}$ which are eigenstates of S , we have $\dim \mathcal{F}(\Psi) \leq 4$, and a basis of $\mathcal{F}(\Psi)$ is one of the following ten sets:

$$\begin{aligned} & \{\Psi, Q^\dagger \Psi, Q^{\dagger 2} \Psi, Q^{\dagger 3} \Psi\}, \{Q \Psi, \Psi, Q^\dagger \Psi, Q^{\dagger 2} \Psi\}, \{Q^2 \Psi, Q \Psi, \Psi, Q^\dagger \Psi\}, \\ & \{Q^3 \Psi, Q^2 \Psi, Q \Psi, \Psi\}, \{\Psi, Q^\dagger \Psi, Q^{\dagger 2} \Psi\}, \{Q \Psi, \Psi, Q^\dagger \Psi\}, \\ & \{Q^2 \Psi, Q \Psi, \Psi\}, \{\Psi, Q^\dagger \Psi\}, \{\Psi, Q \Psi\}, \{\Psi\}. \end{aligned}$$

Proof: First note that any state ϕ which includes both Q 's and Q^\dagger 's acting on Ψ can be written as a scalar multiple of some state with only Q 's or only Q^\dagger 's acting on Ψ . Indeed, we would certainly have Q and Q^\dagger as neighbours somewhere, i.e. $\phi = X(QQ^\dagger)Y\Psi$ or $\phi = X(Q^\dagger Q)Y\Psi$, where X and Y represent some products of Q 's and Q^\dagger 's. But because of Proposition 2, we have that $Y\Psi$ is an eigenstate of QQ^\dagger and so $\phi = c \cdot XY\Psi$. By continuing this process we can eliminate either all Q 's or all Q^\dagger 's (whichever were less in ϕ). Furthermore, it is obvious that $Q^4\Psi = Q^{\dagger 4}\Psi = 0, \forall \Psi \in \mathcal{H}$, because of the stratification of \mathcal{H} , and so we obtain the first four cases and the fact that $\dim \mathcal{F}(\Psi) \leq 4$. If, in addition, we have $Q^{\dagger 3}\Psi = 0$ or $Q^{\dagger 2}\Psi = 0$ or $Q^\dagger \Psi = 0$ or $Q\Psi = 0$ or $Q^2\Psi = 0$ or $Q^3\Psi = 0$, we obtain the remaining “degenerate” cases. ■

Definition 3 The family $\mathcal{F}(\Psi)$ is called generic if $\dim \mathcal{F}(\Psi) = 4$.

Since Ψ is an eigenstate of N we have that all four basis vectors of a generic family belong to different subspaces \mathcal{H}_j .

Proposition 4 If Ψ_i and Ψ_j are eigenstates of S , and if $\Psi_i \notin \mathcal{F}(\Psi_j)$ then $\mathcal{F}(\Psi_i) \cap \mathcal{F}(\Psi_j) = \{0\}$.

Proof: We just have to prove that two other elements of $\mathcal{F}(\Psi_i)$, $Q\Psi_i$ and $Q^\dagger \Psi_i$, do not belong to $\mathcal{F}(\Psi_j)$. Indeed, if $Q\Psi_i \neq 0$, then $Q^\dagger Q\Psi_i = c^2\Psi_i$ with $c \neq 0$ and so if $Q\Psi_i \in \mathcal{F}(\Psi_j) \setminus \{0\}$ then $\Psi_i = \frac{1}{c^2} Q^\dagger \underbrace{Q\Psi_i}_{\in \mathcal{F}(\Psi_j)} \in \mathcal{F}(\Psi_j)$ which is a contradiction. ■

Now, completely analogously to the cases $p = 1$ and $p = 2$ (see e.g. [6]) we have the following theorem:

Theorem 1 Let (\mathcal{H}, H) be a $p = 3$ parasupersymmetric pair and let the spectrum of H be bounded below on \mathcal{H} with a finite multiplicity for each eigenvalue. Then there exists a sequence $\Psi_n \in \mathcal{H}$ such that

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{F}(\Psi_n) .$$

4 A generalized Witten index

The Witten index in supersymmetric quantum mechanics [8] is given by $(\mathcal{P}_1 =) \text{Tr}(-1)^F$ (as we said in section 2, in this case $F = N$), and in [6] it was generalized to all p . In particular in the $p = 2$ parasupersymmetric case we have $\mathcal{P}_2 = \text{Tr}(e^{2\pi Ni/3})$ and for $p = 3$ we have the following:

Definition 4 Given a $p = 3$ parasupersymmetric pair satisfying the conditions of Theorem 1, and such that Q is Fredholm, its parasupersymmetry index \mathcal{P}_3 is given by

$$\mathcal{P}_3 = \text{Tr} \left(e^{\frac{2\pi i}{4}} \right)^N = \text{Tr} i^N .$$

As in the cases of lower p , we can find an expression for the parasupersymmetry index as a certain type of analytical index, in an analogous fashion to index theorems.

Theorem 2 If we define the analytical index of the “paracomplex” C :

$$0 \xleftarrow{Q^{(0)}} \mathcal{H}_0 \xrightleftharpoons[Q^{(1)}]{Q^{\dagger(0)}} \mathcal{H}_1 \xrightleftharpoons[Q^{(2)}]{Q^{\dagger(1)}} \mathcal{H}_2 \xrightleftharpoons[Q^{(3)}]{Q^{\dagger(2)}} \mathcal{H}_3 \xrightarrow{Q^{\dagger(3)}} 0$$

where $Q^{(i)} = Q|_{\mathcal{H}_i}$, $Q^{\dagger(i)} = Q^{\dagger}|_{\mathcal{H}_i}$, to be:

$$\begin{aligned} \text{ind}_A C &= \dim \ker Q^{\dagger(0)} + (1 + i) \dim \ker Q^{\dagger(1)} + i \dim \ker Q^{\dagger(2)} \\ &\quad - \dim \ker Q^{(1)} - (i + 1) \dim \ker Q^{(2)} - i \dim \ker Q^{(3)}, \end{aligned}$$

then:

$$\text{ind}_A C = \mathcal{P}_3. \tag{6}$$

Proof: Because of Theorem 1, \mathcal{H} decomposes as a sum of families $\mathcal{F}(\Psi)$ and so \mathcal{P}_3 is a sum of traces of i^N on $\mathcal{F}(\Psi)$. On generic families the trace obviously vanishes since we have one basis vector in each \mathcal{H}_i and so $\text{Tr} i^N|_{\mathcal{F}(\Psi)} = i^0 + i^1 + i^2 + i^3 = 0$. Thus we only have to show the equality on all types of nongeneric families, i.e. on the corresponding basis elements, which have Ψ 's from different subspaces \mathcal{H}_j . (The non-generic families are finite in number because of the Fredholm condition on Q .) This is a simple case-by-case check, for instance, for the cases $\{\Psi_0, (Q^{\dagger}\Psi)_1, (Q^{\dagger 2}\Psi)_2\}$, $\{(Q\Psi)_0, \Psi_1, (Q^{\dagger}\Psi)_2\}$ or $\{(Q^2\Psi)_0, (Q\Psi)_1, \Psi_2\}$, where the subscript j denotes that the state belongs to \mathcal{H}_j , we obviously have a contribution of $1 + i + i^2 = i$ to \mathcal{P}_3 and 1 to $\dim \ker Q^{\dagger(2)}$. In other cases we work analogously. ■

5 A model with $p = 3$ parasupersymmetry

By analogy with the $p = 1$ and $p = 2$ models from section 2, we can introduce a class of models with one bosonic and three parafermionic levels, which are parasupersymmetric of order $p = 3$. By a direct generalization, we can introduce parasupersymmetry operators acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^4)$ as follows:

$$Q \begin{pmatrix} f_3 \\ f_2 \\ f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} 0 \\ (-d_x + W_1)f_3 \\ (-d_x + W_2)f_2 \\ (-d_x + W_3)f_1 \end{pmatrix}, \quad Q^{\dagger} \begin{pmatrix} f_3 \\ f_2 \\ f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} (d_x + W_1)f_2 \\ (d_x + W_2)f_1 \\ (d_x + W_3)f_0 \\ 0 \end{pmatrix} \tag{7}$$

where the $W_i(x)$'s are superpotentials whose properties are yet to be specified. In order for the operators to satisfy relation (iii) of our parasusy algebra, the superpotentials must satisfy:

$$W_2^2 - W_1^2 + W_2' + W_1' = c \quad (8)$$

$$W_3^2 - W_2^2 + W_3' + W_2' = d, \quad (9)$$

with c and d being arbitrary constants. Since we want our Hamiltonian to be of the form:

$$6H = (-3d_x^2)\mathbb{I}_4 + \text{diag}(h_1, h_2, h_3, h_4)$$

for some functions h_i , the condition that the parasusy operators commute with H (relation (i) of the parasusy algebra) is equivalent to the following relations for the h_i 's:

$$h_i = 6W_i' + h_{i+1}, \quad i = 1, 2, 3, \quad (10)$$

$$h_1' = 3(W_1' + W_1^2)'. \quad (11)$$

By using (8) and (9), we can obtain the following form for h_1 :

$$h_1 = W_1^2 + W_2^2 + W_3^2 + 5W_1' + 3W_2' + W_3' + e, \quad (12)$$

with e being a new constant. For the other h_i 's we find similar expressions by using the relation (10).

By direct calculations, one can show that neither bilinear nor trilinear combinations of parasupercharges give appropriate products of parasupercharges and Hamiltonians (analogously to the $p = 1$ and $p = 2$ parasusy algebras). However, their quadrilinear combinations satisfy exactly the remaining relations (ii) and (iv) from our $p = 3$ parasusy algebra, provided that $e = 0$. Indeed, in order to satisfy (ii) (and thus (iv) since it is the “adjoint” of (ii)), we need that $e = 0$. Thus we have obtained a model where the parasupercharges are given by (7), with the superpotentials satisfying

$$\begin{aligned} (W_2^2 - W_1^2)' + W_2'' + W_1'' &= 0 \\ (W_3^2 - W_2^2)' + W_3'' + W_2'' &= 0, \end{aligned}$$

and with the Hamiltonian H given by

$$H = \left(-\frac{1}{2}d_x^2 + \frac{1}{6}(W_1^2 + W_2^2 + W_3^2) \right) \mathbb{I}_4 + \text{diag}(U_1, U_2, U_3, U_4) \quad (13)$$

where

$$\begin{aligned} U_1 &= 5W_1' + 3W_2' + W_3' \\ U_2 &= -W_1' + 3W_2' + W_3' \\ U_3 &= -W_1' - 3W_2' + W_3' \\ U_4 &= -W_1' - 3W_2' - 5W_3' \end{aligned}$$

The Hamiltonian is obviously diagonal and we can define \mathcal{H}_0 , \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 as the subspaces given by $\{f_3 = f_2 = f_1 = 0\}$, $\{f_3 = f_2 = f_0 = 0\}$, $\{f_3 = f_1 = f_0 = 0\}$, $\{f_2 = f_1 = f_0 = 0\}$, respectively. In this way we obtain a $p = 3$ parasupersymmetric pair (\mathcal{H}, H) . The parafermion number operator in this model is given by $N = \text{diag}(3, 2, 1, 0)$. Note that if we multiply Q by i and Q^\dagger by $-i$, we obtain the physics convention form of these operators (like in [7]).

One of the physical interpretations of this model is that, for a specific choice of superpotentials, we can obtain a Hamiltonian which describes the one-dimensional motion of a spin $3/2$ particle in a magnetic field directed along the third axis:

$$H = -\frac{1}{2}d_x^2 + V(x) + B(x)J_3, \quad (14)$$

where J_3 is the diagonal generator of the four-dimensional representation of $\text{SO}(3)$, i.e. $J_3 = N - 3/2$. In order that the forms (13) and (14) of the Hamiltonian coincide, we must have that $W_1' = W_2' = W_3' = B(x)$ and that the superpotentials have the following forms:

$$\begin{aligned} W_3(x) &= W_2(x) + \alpha = W_1(x) + 2\alpha \\ W_1(x) &= \begin{cases} k_1x + k_2, & \alpha = 0 \\ k_1e^{-\alpha x} + k_2, & \alpha \neq 0 \end{cases} \end{aligned}$$

with α, k_1, k_2 being arbitrary constants. The potential term is in all cases given by

$$V(x) = (W_1^2 + W_2^2 + W_3^2)/6,$$

and so we see that the case $\alpha = 0$ corresponds to a harmonic oscillator in a homogeneous magnetic field, and the case $\alpha \neq 0$ corresponds to the Morse potential and exponential magnetic field. This is in agreement with the discussion in [7].

6 Spectra and the index

As in [7] in the $p = 2$ case, we can introduce the hermitian parasusy charges:

$$Q_1 = \frac{Q^\dagger + Q}{2\sqrt{3}}, \quad Q_2 = \frac{Q^\dagger - Q}{2\sqrt{3}i}.$$

In terms of Q_1 , Q_2 and H , our $p = 3$ parasusy algebra from Definition 1 becomes:

$$[Q_1, H] = [Q_2, H] = 0 \quad (15)$$

$$\Sigma^{1,3} = \Sigma^{3,1} \quad (16)$$

$$\Sigma^{4,0} + \Sigma^{0,4} = \Sigma^{2,2} \quad (17)$$

$$\Sigma^{1,3} + \Sigma^{3,1} = \Sigma^{1,1}H \quad (18)$$

$$2(Q_1^4 - Q_2^4) = (Q_1^2 - Q_2^2)H \quad (19)$$

$$Q_1^3Q_2 + Q_2^2Q_1Q_2 + Q_2Q_1Q_2^2 + Q_2Q_1^3 = Q_2^3Q_1 + Q_1^2Q_2Q_1 + Q_1Q_2Q_1^2 + Q_1Q_2^3, \quad (20)$$

where $\Sigma^{i,j}$ denotes the sum of all monomials with i Q_1 's and j Q_2 's. We may use this form of the parasusy algebra to show that the eigenvalues of H are generically non-negative. Since Q_1 and H commute we can diagonalize them simultaneously. Let Ψ be a simultaneous eigenstate of Q_1 and H with eigenvalues q_1 and E respectively. Then from equation (19) we have the equation

$$2(q_1^2 + Q_2^2)(q_1^2 - Q_2^2)\Psi = 2(q_1^4 - Q_2^4)\Psi = (q_1^2 - Q_2^2)E\Psi.$$

Thus if $(q_1^2 - Q_2^2)\Psi \neq 0$ we deduce that E is non-negative, since it is an eigenvalue of the positive operator $2(q_1^2 + Q_2^2)$. Therefore the only states which may have negative energy eigenvalues are those belonging to $\ker(Q_2^2 - q_1^2)$. Let us assume that Ψ belongs to this kernel. Furthermore, from (18) we obtain:

$$(Q_1^2 + Q_2^2)(Q_1Q_2 + Q_2Q_1)\Psi + (Q_1Q_2 + Q_2Q_1)(Q_1^2 + Q_2^2)\Psi = (Q_1Q_2 + Q_2Q_1)E\Psi$$

which gives:

$$(Q_1^2 + Q_2^2)(Q_1Q_2 + Q_2Q_1)\Psi = (E - 2q_1^2)(Q_1Q_2 + Q_2Q_1)\Psi. \quad (21)$$

So, if $(Q_1Q_2 + Q_2Q_1)\Psi \neq 0$ then $E - 2q_1^2$ (and hence E) must be nonnegative since it is an eigenvalue of the positive operator $(Q_1^2 + Q_2^2)$. So, suppose that $(Q_1Q_2 + Q_2Q_1)\Psi = 0$. In this case, we see that all relations (15)-(20) of our parasusy algebra are satisfied. Now if we go back to Q and Q^\dagger , we obviously have:

$$Q^2\Psi = 3(Q_1 - iQ_2)^2\Psi = 3(Q_1^2 - Q_2^2)\Psi - 3i(Q_1Q_2 + Q_2Q_1)\Psi = 0, \quad (22)$$

and analogously

$$Q^{\dagger 2}\Psi = 0. \quad (23)$$

Furthermore, since Ψ is an eigenfunction of Q_1 , we have

$$Q\Psi + Q^\dagger\Psi = c\Psi, \quad (24)$$

for some real c . We conclude that the state Ψ can have negative energy, if and only if it satisfies (22), (23) and (24). Thus, unlike in the $p = 1$ supersymmetric case, where the energy spectrum is positive definite (see [8]), there is no restriction on the parasupersymmetric vacuum energy in the $p = 3$ case (in the same way as in the $p = 2$ case - see [7]).

Because of the conservation of parasupercharges, and because of the 4-fold stratification of \mathcal{H} , we know that the spectrum of H is generically 4-fold degenerate, except for a few lowest energy levels. Indeed, since N and H commute, we can diagonalize them simultaneously. Then because of $[N, Q^\dagger] = Q^\dagger$, $Q^{\dagger 4} = 0$ and since $[Q^\dagger, H] = 0$, any state ϕ with zero parafermions ($N = 0$), and the states $Q^\dagger\phi$, $Q^{\dagger 2}\phi$ and $Q^{\dagger 3}\phi$, have the equal energy (eigenvalue of Hamiltonian). So, obviously, in the generic case, for each eigenvalue, we would have four linearly independent eigenstates (one from each \mathcal{H}_j). The only way to have less than 4-fold degeneracy is when the family generated by Ψ (i.e. $\mathcal{F}(\Psi)$) is nongeneric. Thus, we have to look at $\ker Q$ and $\ker Q^\dagger$.

Our goal is to connect the possible forms of the spectrum of H from our class of models from Section 5, with the corresponding index. By using theorem 2 (i.e. formula (6)), we see that for calculating the index we need to know the corresponding dimensions of the kernels of the restrictions of the parasupercharges on different \mathcal{H}_i 's. For instance, the only way for $\dim \ker Q^{(3)}$ to be non-zero is when the vector $(\Psi_+^{(1)}, 0, 0, 0)^T$ is normalizable in $L^2(\mathbb{R}, \mathbb{R}^4)$, where

$$\Psi_{\pm}^{(i)} = \exp\left(\pm \int^x W_i(t) dt\right), \quad i = 1, 2, 3.$$

When this vector is normalizable, we obviously have that the dimension of the kernel of the corresponding operator is equal to 1. Thus we have:

$$\begin{aligned} \dim \ker Q^{(3)} = 1 &\iff \Psi_+^{(1)} \in L^2(\mathbb{R}, \mathbb{R}) \\ \dim \ker Q^{\dagger(2)} = 1 &\iff \Psi_-^{(1)} \in L^2(\mathbb{R}, \mathbb{R}) \\ \dim \ker Q^{(2)} = 1 &\iff \Psi_+^{(2)} \in L^2(\mathbb{R}, \mathbb{R}) \\ \dim \ker Q^{\dagger(1)} = 1 &\iff \Psi_-^{(2)} \in L^2(\mathbb{R}, \mathbb{R}) \\ \dim \ker Q^{(1)} = 1 &\iff \Psi_+^{(3)} \in L^2(\mathbb{R}, \mathbb{R}) \\ \dim \ker Q^{\dagger(0)} = 1 &\iff \Psi_-^{(3)} \in L^2(\mathbb{R}, \mathbb{R}), \end{aligned}$$

with the dimensions of the kernels being equal to 0 in all other cases.

Thus the spectrum of H and the index for our model depend on the possible nontrivial kernels of the parasupercharges (and their restrictions), which in turn depend on the square integrability of the functions $\Psi_{\pm}^{(i)}$. Since, for each i we have that either $\Psi_+^{(i)}$ is normalizable or $\Psi_-^{(i)}$ is normalizable or neither of them is, we can theoretically have $3^3 = 27$ different outcomes for the spectrum and the value of the index. One can calculate the value of the index and sketch the form of the spectrum in all possible cases. One can also find explicit examples for W_i corresponding to all these cases. In the cases of lower p , by repeating the same procedure, one can obtain a one-to-one correspondence between the spectrum and the index (see [7] for an analysis of the spectrum in $p = 2$ case).

However in the $p = 3$ case there are only 19 different outcomes for the value of the index, so the forms of the spectrum will also depend on one more parameter - namely the sign of the constants c and d from equations (8) and (9). Furthermore, the constant c , together with one more constant c_0 and a function K classify all possible solutions of equation (5) (identical to equation (8)), which is the only restriction in the $p = 2$ case ([7], [6]). Rewriting equation (8) as follows:

$$(W_1 + W_2)' + (W_1 + W_2)(W_1 - W_2) = c,$$

it becomes a linear differential equation for the function $W_1 + W_2$. If we denote

$$K = \int^x (W_2 - W_1),$$

its solution is given by the formula:

$$W_1 + W_2 = e^{-K} \left(c_0 + c \int_0^x e^K \right), \quad c_0 \in \mathbb{R},$$

which together with $W_1 - W_2 = -K'$, gives:

$$W_{1,2} = \frac{1}{2} \left(\mp K' + c_0 e^{-K} + c e^{-K} \int_0^x e^K \right). \quad (25)$$

Thus, in the case $p = 2$, the class of models is characterized by one function (K) and two constants (c, c_0). In the $p = 3$ case we have to go on and solve equation (9), where now W_2 is a known function (depending on K, c, c_0), and so we just have a first-order equation (the Riccati equation) with respect to W_3 . This will give us one new constant d_0 , which, together with the existing d , shows that our class of models is characterized by one function and four constants.

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